Backpaper - Probability II (2025-26)
Time: 3 hours.
Attempt all questions.

1. Consider a simple symmetric random walk S_n on $\mathbf{Z}$ with probability of jumping to the right and left both equal to $\frac{1}{2}$. Starting at $S_0 = 0$ what is the probability that the walk does not hit 3 up to time 100 and $S_{100} = 50$? **[5 marks]**

2. Consider a simple random walk on $\mathbf{Z}$ starting at the point 0, with probability of going to the right equal to $\frac{3}{4}$ and going to the left equal to $\frac{1}{4}$. Find the probability that the walk hits the point -100 before the point 100. **[5 marks]**

3. The joint probability density function of X and Y is given by

$$f(x,y) = \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) \qquad 0 < x < 1,\ 0 < y < 2.$$

   (a) Verify that this is indeed a joint density function. **[2 marks]**
   (b) Compute the density function of X. **[2 marks]**
   (c) Find $P(X > Y)$ **[4 marks]**
   (d) Find $E(Y|X = x)$. **[4 marks]**
   (e) Find $E(Y)$. **[3 marks]**

4. If X and Y are independent and identically distributed uniform random variables on $(0, 1)$ compute the joint density of $U = X + Y,\ V = X/Y$. **[5 marks]**

5. Let (Ω, P) be a discrete probability space, that is Ω has finite or countably many outcomes. Let X and Y be random variables on (Ω, P). Show

$$P(X \geq a) \leq e^{-ta}M(t) \quad \text{for all } t > 0,$$

   where $M(t) = E[e^{tX}]$ is the *moment generating function* of X. **[5 marks]**

6. Let X_n, Y_n, X, Y be random variables on a common probability space. If $X_n \xrightarrow{P} X$ and $Y_n \xrightarrow{P} Y$ show that $X_n + Y_n \xrightarrow{P} X + Y$. **[5 marks]**

7. Let $X_1, X_2, \cdots, X_n$ be a set of i.i.d. continuous random variables with CDF F and density f, and let $X_{(1)} < X_{(2)} < \cdots < X_{(n)}$ denote their ordered values. Compute the CDF and pdf of $X_{(n)}$. **[5 marks]**

8. Let $X_1, X_2, \cdots, X_{1000}$ be i.i.d. Bernoulli(0.4) random variables, that is $X_i = 1$ with probability 0.4 and $X_i = 0$ with probability 0.6. Use the central limit theorem to give an approximation of

$$P\left(200 \leq \sum_{i=1}^{1000} X_i \leq 500\right)$$

   in terms of the CDF Φ of the standard normal distribution. **[5 marks]**